数理解析研究所講究録 1412 巻 2005 年 110-124

Distributions of exponential growth with support in a proper convex cone

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1 Introduction

In this talk we treated the space $H'(\mathbb{R}^n, K)$ of distributions of exponential growth. The spaces of distributions of exponential growth for the 1-dimensional case, direct product case or global case were investigated by many authors ([3], [5], [6], [10], [11], [12], [13], [15], [17]). In [3] M.Hasumi studied the space $H(\mathbb{R}^n, \mathbb{R}^n)$ and the dual space $H'(\mathbb{R}^n, \mathbb{R}^n)$ (see Definition 3.2 and Definition 3.5). In [10] M.Morimoto studied the space $H(\mathbb{R}^n, K)$ and the dual space $H'(\mathbb{R}^n, K)$ (see Definition 3.2 and Definition 3.5). The purpose of this talk was to treat the space of distributions of exponential growth *supported by a proper convex cone* $\overline{\Gamma} \subset \mathbb{R}^n$, (denote by $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$).

In §3 we shall state the base space $H(\mathbb{R}^n, K)$ and its dual space $H'(\mathbb{R}^n, K)$. The main purpose in this section is to introduce the structure theorem for $H'_{\overline{A}}(\mathbb{R}^n, K)$, the space of distributions of exponential growth *supported by a set* $\overline{A} \subset \mathbb{R}^n$ (Theorem 3.7). Therefore as corollary we obtain the structure theorem for $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$, where $\overline{\Gamma} \subset \mathbb{R}^n$ is a proper convex cone, (Corollary 3.8), and the result which G.Lysik obtained for the case of direct product support of half lines ([6]). Furthermore we have the decomposition theorem for distributions of exponential growth with support in $\overline{\Gamma}_+ \cup \overline{\Gamma}_-$, (Corollary 3.10).

In §4 we shall characterize the space $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ by the heat kernel method (Theorem 4.1), which T.Matsuzawa introduced for the spaces of distributions, ultradistributions and hyperfunctions [2], [7], [8], [9].

In §5 we shall introduce the Paley-Wiener theorem for $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. Then we showed that the Fourier-Laplace transform of $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ is a holomorphic function constructed by a finite sum of functions which are holomorphic on the domains whose imaginary parts are proper convex cones with

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vertex at the elements of K and with some polynomial growth conditions and conversely such a holomorphic function can be represented by the Fourier-Laplace transform of a distribution of exponential growth $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. Then we can see that T is constructed by a finite sum of distributions of exponential growth supported by a proper convex cone $\overline{\Gamma}$ (Theorem 5.5). As corollary we have the result which M.Morimoto showed for the 1-dimensional case [10].

In §6 we shall state the space of the image by the Fourier-Laplace transform of $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. Then by using the Paley-Wiener theorem given in §5, we can obtain the Edge-of-the-Wedge theorem for this space (Theorem 6.10). These results are generalizations of the work which M.Morimoto showed for the case of direct product ([11], Theorem 2).

2 Preliminaries

Definition 2.1. We define some notations:

$$\begin{aligned}
x &= (x_1, \cdots, x_n) \in \mathbb{R}^n,\ \langle x, \xi\rangle = \sum_{j=1}^{n} x_j \xi_j \quad \text{for } x, \xi \in \mathbb{R}^n,\ x^2 = \langle x, x\rangle. \\
z &= (z_1, \cdots, z_n) \in \mathbb{C}^n, \quad z_j = x_j + iy_j, \quad j = 1, \cdots, n. \\
\alpha &= (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n,\ D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \quad \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \\
E(x,t) &= (4\pi t)^{-\frac{n}{2}} \exp(-x^2/4t), \quad t > 0.
\end{aligned}$$

For $\zeta \in \mathbb{C}^n$, $\zeta = (\zeta_1, \cdots, \zeta_n)$, we put $|\zeta| = \sqrt{|\zeta_1|^2 + \cdots + |\zeta_n|^2}$.

Definition 2.2. Let K be a convex compact set in $\mathbb{R}^n$. Then we define supporting function of K by $h_K(x) = \sup_{\xi \in K} \langle x, \xi\rangle$.

Definition 2.3. Let Ω be an open set in $\mathbb{C}^n$. We denote by $\mathcal{H}(\Omega)$ the space of holomorphic functions on Ω and by $\mathcal{C}(\Omega)$ the space of continuous functions on Ω.

Definition 2.4. $\mathcal{S}(\mathbb{R}^n)$ is the space of rapidly decreasing $\mathcal{C}^\infty$ functions and $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions.

Definition 2.5. Let A be a set in $\mathbb{R}^n$. Then we denote by A° the interior of A, $\overline{A}$ the closure of A, for $\varepsilon > 0$, $A_\varepsilon = \{x \in \mathbb{R}^n; \mathrm{dis}(x, A) \leq \varepsilon\}$ and by $\mathrm{ch}(A)$ convex hull of A.

Definition 2.6. Let Γ be a cone with vertex at 0. If $\overline{\mathrm{ch}\Gamma}$ contains no straight line, then we call Γ proper cone.

Definition 2.7 **([4],[16]).** Let Γ be a cone. We put

$$\Gamma' := \{\xi \in \mathbb{R}^n; \langle y, \xi \rangle \geq 0 \text{ for all } y \in \Gamma\}.$$

Then we call Γ' dual cone of Γ.

Definition 2.8. Let Γ be a cone. Then we denote by $\mathrm{pr}\Gamma$ the intersection of Γ and the unit sphere. The cone Γ_1 is said to be a compact cone in the cone Γ_2 if $\mathrm{pr}\overline{\Gamma}_1 \subset \mathrm{pr}\Gamma_2$ and we write $\Gamma_1 \Subset \Gamma_2$.

Proposition 2.9 **([16]).** *Following conditions are equivalent:*

1. Γ *is proper cone.*
2. $(\Gamma')^\circ \neq \emptyset$.
3. *For any* $C \Subset (\Gamma')^\circ$, *there exists a number* $\sigma = \sigma(C) > 0$ *such that* $\langle \xi, x \rangle \geq \sigma |\xi||x|$, $\xi \in C$, $x \in \mathrm{ch}\overline{\Gamma}$.

Proposition 2.10 **([16]).** $(\Gamma')' = \overline{\mathrm{ch}\Gamma}$ *and* $(\Gamma_1 \cap \Gamma_2)' = \mathrm{ch}({\Gamma'}_1 \cup {\Gamma'}_2)$. *Furthermore for a convex cone* Γ, *we have* $\Gamma = \Gamma + \Gamma$.

Definition 2.11. Let Γ_+ be a cone with vertex at 0. Then we put $\Gamma_- = -\Gamma_+$.

Definition 2.12. Let A be a set in $\mathbb{R}^n$. We put $\mathcal{S}'_{\overline{A}} := \{T \in \mathcal{S}'(\mathbb{R}^n); \mathrm{supp}\, T \subset \overline{A}\}$.

3 Distributions of exponential growth

In this section, we shall introduce $H'(\mathbb{R}^n, K)$, the space of distributions of exponential growth, and give the structure theorem of $H'(\mathbb{R}^n, K)$.

Definition 3.1. Let K be a convex compact set in $\mathbb{R}^n$ and $\varepsilon > 0$. Then we define $H_b(\mathbb{R}^n, K_\varepsilon)$ as follows:

$$H_b(\mathbb{R}^n, K_\varepsilon) \quad := \quad \{\varphi \in C^\infty(\mathbb{R}^n); \sup_{x\in\mathbb{R}^n} |D^p\varphi(x)e^{h_K(x)+\varepsilon|x|}| < +\infty, \text{for } \forall p \in \mathbb{N}^n\}.$$

Definition 3.2. We define the spaces $H(\mathbb{R}^n, \mathbb{R}^n)$ and $H(\mathbb{R}^n, K)$ as follows:

$$H(\mathbb{R}^n, \mathbb{R}^n) := \varprojlim_{\varepsilon>0} H_b(\mathbb{R}^n, K_\varepsilon), \quad H(\mathbb{R}^n, K) := \varinjlim_{\varepsilon>0} H_b(\mathbb{R}^n, K_\varepsilon),$$

where $\varprojlim_{\varepsilon>0}$ means projective limit and $\varinjlim_{\varepsilon>0}$ means inductive limit.

Remark 3.3. Now we give the relations of $H(\mathbb{R}^n, K)$ and the other function spaces:

(i) If $\{0\} \subset K$, then $H(\mathbb{R}^n, K) \subset \mathcal{S}$.

(ii) Let $r \geq 0$, $s \geq 0$, $\mathcal{S}_r^s(\mathbb{R}^n)$ be Gel'fand-Shilov space and $\mathcal{S}_r(\mathbb{R}^n) = \varinjlim_{s\to\infty} \mathcal{S}_r^s(\mathbb{R}^n)$. Then it is known that

$$\mathcal{S}_1(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n); \exists\delta > 0\ \forall\alpha \sup_{x\in\mathbb{R}^n} |D_x^\alpha f(x)|e^{\delta|x|} < \infty\},$$

(for details we refer the reader [12]). Therefore

(a) If $K = \{0\}$, then $H(\mathbb{R}^n, K) = \mathcal{S}_1(\mathbb{R}^n)$.

(b) If $\{0\} \subset K$, then $H(\mathbb{R}^n, K) \subset \mathcal{S}_1(\mathbb{R}^n)$.

(iii) The space $H(\mathbb{R}^n, K)$ is slightly different from $\mathfrak{F}_E$ in [1]. In fact

$$\varphi(x) \in H(\mathbb{R}^n, K) \Leftrightarrow \exists\varepsilon > 0\ \forall p \in \mathbb{N}^n \ s.t. \sup_{x\in\mathbb{R}^n} |D^p\varphi(x)e^{h_K(x)+\varepsilon|x|}| < \infty.$$

$$\varphi(x) \in \mathfrak{F}_E \Leftrightarrow \forall p \in \mathbb{N}^n\ \exists k > 0 \ s.t. \sup_{x\in\mathbb{R}^n} |D^p\varphi(x)|e^{k|x|} < \infty.$$

Therefore if $\{0\} \subset K$, then $H(\mathbb{R}^n, K) \subset \mathfrak{F}_E$.

Remark 3.4. L.Hörmander treated the base space $\mathcal{S}_f$ so that $\mathcal{D} \subset \mathcal{S}_f \subset H(\mathbb{R}^n, K)$ and the Fourier-Laplace transform of $\mathcal{S}_f$. For the details we refer the reader to [5].

Definition 3.5. We denote by $H'(\mathbb{R}^n, \mathbb{R}^n)$ the dual space of $H(\mathbb{R}^n, \mathbb{R}^n)$ and by $H'(\mathbb{R}^n, K)$ the dual space of $H(\mathbb{R}^n, K)$. The elements of $H'(\mathbb{R}^n, \mathbb{R}^n)$ and $H'(\mathbb{R}^n, K)$ are called distributions of exponential growth.

Definition 3.6. We put $H'_{\overline{A}}(\mathbb{R}^n, K) := \{T \in H'(\mathbb{R}^n, K); \operatorname{supp} T \subset \overline{A}\}$.

Now we have the structure theorem for distributions of exponential growth with support $\overline{A} \subset \mathbb{R}^n$:

Theorem 3.7 ([14]). *Let A be a set in $\mathbb{R}^n$ and $T \in H'_{\overline{A}}(\mathbb{R}^n, K)$. Then for every $\varepsilon > 0$ there exist $S(x) \in \mathcal{S}'_{\overline{A}}$, $n_0 \in \mathbb{N}$ and $t_j \in K$, $j = 1, 2, \cdots, n_0$ such that*

$$T = S(x) e^{\varepsilon\sqrt{1+x^2}} \sum_{1 \le j \le n_0} e^{t_j x}.$$

For $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$, we have the following corollaries:

Corollary 3.8 ([14]). *Let Γ be a proper open convex cone in $\mathbb{R}^n$ and let $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. Then for any $\varepsilon > 0$ there exist $m_\varepsilon \in \mathbb{N}$ and bounded continuous functions $F_{\varepsilon,\alpha}(x)$, $|\alpha| \le m_\varepsilon$,* $\operatorname{supp}(F_{\varepsilon,\alpha}(x)) \subset \overline{\Gamma}$ *such that*

$$T = \sum_{|\alpha| \le m_\varepsilon} \left(\frac{\partial}{\partial x}\right)^\alpha \{e^{h_K(x) + \varepsilon|x|} F_{\varepsilon,\alpha}(x)\}.$$

Corollary 3.9 ([14]). *Let Γ be a proper open convex cone in $\mathbb{R}^n$ and let $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. Then for any $\varepsilon > 0$ there exist n_0, a partial differential operator with finite order $P_\varepsilon(D)$ and a polynomially bounded continuous function $G_\varepsilon(x)$,* supp $(G_\varepsilon(x)) \subset \overline{\Gamma}$ *such that*

$$T = P_\varepsilon(D) G_\varepsilon(x) \times F^*(x), \qquad F^*(x) = e^{\varepsilon\sqrt{1+x^2}} \sum_{1 \le n \le n_0} e^{t_n x},$$

where $t_n \in K$, $(n = 1, \cdots, n_0)$.

Corollary 3.10 ([14]). *Let $T \in H'_{\overline{\Gamma}_+ \cup \overline{\Gamma}_-}(\mathbb{R}^n, K)$. Then there exist $T_+ \in H'_{\overline{\Gamma}_+}(\mathbb{R}^n, K)$ and $T_- \in H'_{\overline{\Gamma}_-}(\mathbb{R}^n, K)$ such that*

$$T = T_+ + T_-.$$

Remark 3.11. M.Morimoto obtained this result for the 1-dimensional case in [10].

Example 3.12 (Example for Corollary 3.8). Let $n = 2$, $K = \{(x_1, x_2) \in \mathbb{R}^2; |x| \le 1\}$ and $\Gamma := \{(x_1, x_2) \in \mathbb{R}^2; x_1^2 - x_2^2 > 0, x_1 > 0\}$. We define $T(x)$ by

$$T(x) = \begin{cases} \sqrt{x_1^2 - x_2^2}\, e^{|x|}, & x_1^2 - x_2^2 > 0,\ x_1 > 0, \\ 0, & \textit{otherwise}. \end{cases}$$

Then $h_K(x) = |x|$, $T(x) \in H'_{\overline{\Gamma}}(\mathbb{R}^2, K)$ and for $\varepsilon > 0$,

$$T(x) \;=\; \sqrt{x_1^2 - x_2^2}\, e^{-\varepsilon|x|} e^{|x|} e^{\varepsilon|x|} = F_\varepsilon(x) e^{h_K(x) + \varepsilon|x|},$$

where

$$F_\varepsilon(x) = \begin{cases} \sqrt{x_1^2 - x_2^2}\, e^{-\varepsilon|x|}, & x_1^2 - x_2^2 > 0,\ x_1 > 0, \\ 0, & \textit{otherwise}. \end{cases}$$

Then $F_\varepsilon(x)$ is a bounded continuous function and $\mathrm{supp}(F_\varepsilon) \subset \overline{\Gamma}$.

Example 3.13. Let $n = 1$, $K = \{1\}$ and $\Gamma := (0, \infty)$. We define $T(x)$ by

$$T(x) = \begin{cases} e^x, & x \in \Gamma = (0, \infty), \\ 0, & x \in (-\infty, 0]. \end{cases}$$

Then $T \in H'_{\overline{\Gamma}}(\mathbb{R}, K)$ and for $\varepsilon > 0$

$$T \;=\; \sum_{k=0}^{1} \left(\frac{\partial}{\partial x}\right)^k \{F_{\varepsilon,k}(x) e^{x + \varepsilon|x|}\},$$

where $F_{\varepsilon,k}(x) = (-1)^{k+1} \chi_+(x) e^{-\varepsilon|x|}$ and

$$\chi_+(x) \;=\; \begin{cases} x, & x \in \Gamma = (0, \infty), \\ 0, & x \in (-\infty, 0]. \end{cases}$$

Then $F_{\varepsilon,k}(x)$ is a bounded continuous function and $\mathrm{supp}(F_{\varepsilon,k}) \subset \overline{\Gamma}$.

4 Distributions of exponential growth supported by a proper convex cone

In this section, we shall characterize $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ by the heat kernel method.

Theorem 4.1 ([14]). *Let $\Gamma \subset \mathbb{R}^n$ be a proper open convex cone, $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ and $U(x,t) = \langle T_y, E(x-y,t)\rangle$. Then $U(x,t) \in C^\infty(\mathbb{R}^n \times (0,\infty))$ satisfying the following conditions:*

$$\left(\frac{\partial}{\partial t} - \Delta\right) U(x,t) = 0, \tag{1}$$

$$U(x,t) \to T,\ (t \to 0_+),\ in\ H'(\mathbb{R}^n, K), \tag{2}$$

$$\begin{aligned} &\forall \varepsilon > 0\ \exists N_\varepsilon \geq 0\ \exists C_\varepsilon \geq 0 \\ &s.t.\ |U(x,t)| \leq C_\varepsilon t^{-N_\varepsilon} e^{-\frac{\operatorname{dis}(x,\overline{\Gamma})^2}{16t}} e^{h_K(x)+\varepsilon|x|},\ 0 < t < 1,\ x \in \mathbb{R}^n. \end{aligned} \tag{3}$$

Conversely, for a function $U(x,t) \in C^\infty(\mathbb{R}^n \times (0,\infty))$ satisfying (1) *and* (3)*, there exists a unique $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ such that $\langle T_y, E(x-y,t)\rangle = U(x,t)$.*

Corollary 4.2 ([14]). *Let $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ and $U(x,t) = \langle T_y, E(x-y,t)\rangle$. Then $U(x,t) \in C^\infty(\mathbb{R}^n \times (0,\infty))$ satisfies the following conditions:*

$$\left(\frac{\partial}{\partial t} - \Delta\right) U(x,t) = 0, \tag{4}$$

$$U(x,t) \longrightarrow T,\ (t \to 0_+),\ in\ H'(\mathbb{R}^n, K), \tag{5}$$

$$\begin{aligned} &\forall \varepsilon > 0\ \exists N\ \exists C \geq 0\ s.t.\ |U(x,t)| \leq Ct^{-N} e^{h_K(x)+\varepsilon|x|},\ 0 < t < 1,\ x \in \mathbb{R}^n \\ &and\ U(x,t) \to 0, (t \to 0_+),\ uniformly\ for\ all\ compact\ sets\ in\ \mathbb{R}^n \backslash \overline{\Gamma}. \end{aligned} \tag{6}$$

Conversely, for a function $U(x,t) \in C^\infty(\mathbb{R}^n \times (0,\infty))$ satisfying (4) *and* (6)*, there exists a unique $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ such that $\langle T_y, E(x-y,t)\rangle = U(x,t)$.*

5 Paley-Wiener theorem for $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$

In this section, we shall see the Paley-Wiener theorem for $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. For the 1-dimentional case, it is given in [10].

Definition 5.1. Let Γ be a proper open convex cone, K be a compact set and $\varepsilon' > 0$. Then we denote L by

$$L \ = \ \left\{ \bigcap_{u \in K} (\{u\} + (\overline{\Gamma}')^\circ) \right\}^\circ .$$

Proposition 5.2. $L \neq \emptyset$.

Definition 5.3 ([10], [16]). For $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$, we define the Fourier-Laplace transform $\mathcal{LF}(T)$ of T by

$$\mathcal{LF}(T)(\zeta) \quad := \quad \frac{1}{(2\pi)^{\frac{n}{2}}}\langle T_x, e^{i\zeta x}\rangle, \quad \zeta \in \mathbb{C}^n.$$

The right hand side means

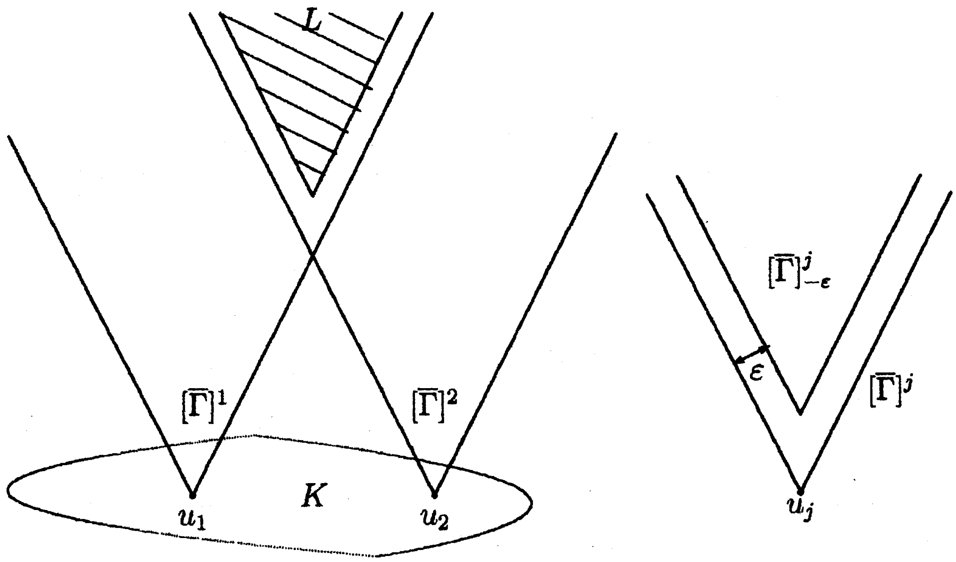
$$\langle T_x, e^{i\zeta x}\rangle \quad = \quad \langle T_x, \chi(x)e^{i\zeta x}\rangle,$$

where $\chi(x) \in \mathcal{C}^\infty(\mathbb{R}^n)$ which satisfies

$$\chi(x) = \begin{cases} 1 & , x \in \overline{\Gamma}_\varepsilon \\ 0 & , x \notin \overline{\Gamma}_{2\varepsilon}, \quad \varepsilon > 0. \end{cases}$$

Definition 5.4. Let Γ be a proper open convex cone and K be a compact set. For $\varepsilon > 0$ and $u_j \in K$, $j = 1, \cdots, j_0$, we set the following notations:

$$[\overline{\Gamma}]^j = (\{u_j\} + \overline{\Gamma})^\circ, \quad [\overline{\Gamma}]^j_{-\varepsilon} = \mathbb{R}^n \backslash (\mathbb{R}^n \backslash [\overline{\Gamma}]^j)_\varepsilon.$$



Theorem 5.5 ([14]). *Let Γ be a proper open convex cone, K be a convex compact set, $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ and $f(\zeta) = \mathcal{LF}(T)(\xi + \imath\eta)$. Then for every $\varepsilon > 0$ there exist $j_0 \in \mathbb{N}$, $l_\varepsilon \geq 0$ and the families $\{u_j\}_{j=1}^{j_0} \subset K$, $\{f_j(\zeta)\}_{j=1}^{j_0}$ satisfying the conditions* (7), (8), (9):

$$f_j(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath[\overline{\Gamma}']^j). \tag{7}$$

$\forall\, \overline{\Gamma}_C \Subset (\overline{\Gamma}')^\circ\ \exists M_{\varepsilon,\overline{\Gamma}_C} \geq 0$ *such that*

$$|f_j(\zeta)| \leq M(1+|\zeta|)^l, \quad \zeta \in \mathbb{R}^n + \imath[\overline{\Gamma}_C]^j_{-2\varepsilon}. \tag{8}$$

$$f(\zeta) = \sum_{1\leq j\leq j_0} f_j(\zeta). \tag{9}$$

In particular, $f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L)$.

Conversely if $f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L)$ satisfies the conditions (7), (8) *and* (9), *then there exists a unique $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ such that $f(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}}\langle T_x, e^{\imath\zeta x}\rangle$. Furthermore T is given by the following formula:*

$$T = \sum_{1\leq j\leq j_0} T_j, \quad T_j \in H'_{\overline{\Gamma}}(\mathbb{R}^n, \{u_j\}), \tag{10}$$

$$f_j(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}}\langle T_{j_x}, e^{\imath\zeta x}\rangle. \tag{11}$$

Corollary 5.6 ([14]). *Let Γ be a proper open convex cone, $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, \{0\})$ and $f(\zeta) = \mathcal{LF}(T)(\xi + \imath\eta)$. Then for $\varepsilon > 0$ there exists $l_\varepsilon \geq 0$ satisfying the conditions* (12), (13):

$$f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L). \tag{12}$$

$\forall \overline{\Gamma}_C \Subset (\overline{\Gamma}')^\circ\ \exists M_{\varepsilon,\overline{\Gamma}_C} \geq 0$ *such that*

$$|f(\zeta)| \leq M(1+|\zeta|)^l, \quad \zeta \in \mathbb{R}^n + \imath[\overline{\Gamma}_C]_{-2\varepsilon}. \tag{13}$$

Conversely if $f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L)$ satisfies the conditions (12) *and* (13), *then there exists a unique $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, \{0\})$ such that $f(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}}\langle T_x, e^{\imath\zeta x}\rangle$.*

Remark 5.7 (Remark for Corollary 5.6). Now we consider more general Fourier-Laplace transforms. That is, if $T \in \mathcal{D}'$ and $e^{-\eta x}T \in \mathcal{S}'$, then we can define the Fourier-Laplace transform $\mathcal{LF}(T)(\zeta)$ of T. Furthermore it is known that we can obtain the Paley-Wiener theorem for $T \in \mathcal{D}'$ if Γ_T° is not empty where $\Gamma_T := \{\eta \in \mathbb{R}^n; e^{-\langle \cdot, \eta\rangle}T \in \mathcal{S}'\}$ (see Theorem 7.4.2 in [4]).

So we can assert that for the Paley-Wiener theorem for $T \in \mathcal{D}'$ (that is, for Theorem 7.4.2 in [4]) we can take the element of the space $H'_{\overline{\Gamma}}(\mathbb{R}^n, \{0\})$ as $T \in \mathcal{D}'$ if and only if the conditions of Corollary 5.6 are satisfied.

Example 5.8 (Example for Theorem 5.5). Let $n = 2$, $K = \{0\}\times[-1,1]$ and $\Gamma := \{x = (x_1, x_2) \in \mathbb{R}^2; x_1^2 - x_2^2 > 0,\ x_1 > 0\}(= (\overline{\Gamma}')^\circ)$. We define $T(x)$ by

$$T(x) = \begin{cases} e^{|x_2|}, & x_1^2 - x_2^2 > 0,\ x_1 > 0, \\ 0, & otherwise. \end{cases}$$

We can see $T \in H'_{\overline{\Gamma}}(\mathbb{R}^2, K)$ and if $\eta \in L := \{\eta = (\eta_1, \eta_2); \{(1,0)\} + (\overline{\Gamma}')^\circ\}$, then

$$\begin{aligned} \langle T_x, e^{\imath\zeta x}\rangle &= \frac{1}{\imath\zeta_1(\imath\zeta_1 + \imath\zeta_2 + 1)} - \frac{1}{\imath\zeta_1(\imath\zeta_1 - \imath\zeta_2 + 1)} \\ &= f_1(\zeta) + f_2(\zeta). \end{aligned}$$

Then we can see $f_1(\zeta) \in \mathcal{H}(\mathbb{R}^2 + \imath L_1)$ and $f_2(\zeta) \in \mathcal{H}(\mathbb{R}^2 + \imath L_2)$, where

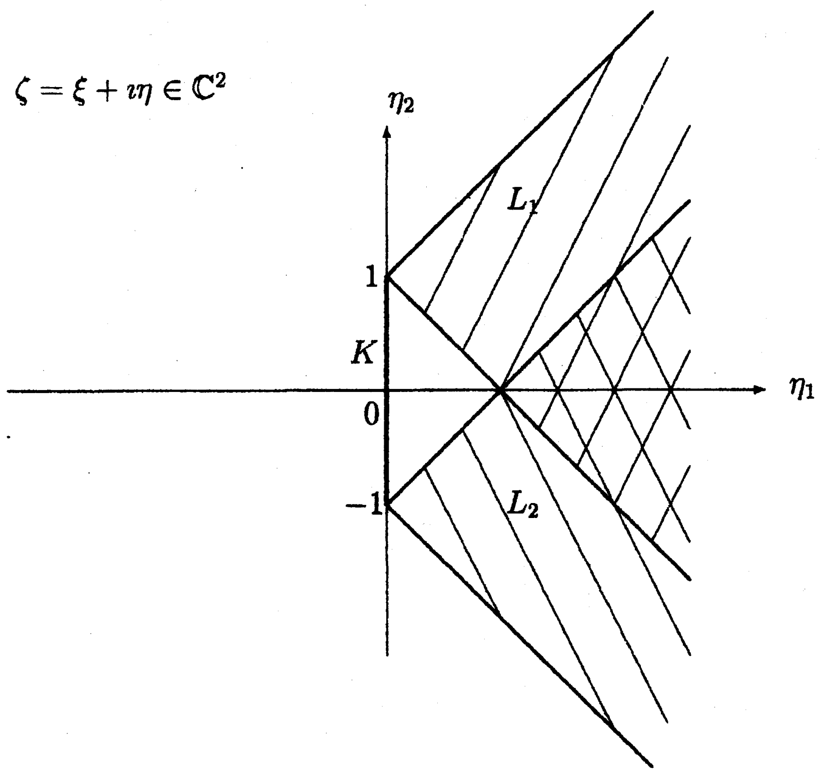
$$L_1 := \{\eta = (\eta_1, \eta_2); \{(0,1)\} + (\overline{\Gamma}')^\circ\}, \quad L_2 := \{\eta = (\eta_1, \eta_2); \{(0,-1)\} + (\overline{\Gamma}')^\circ\},$$

and $L = L_1 \cap L_2$. Now we define

$$T_1 = \begin{cases} e^{x_2}, & x_1 > x_2, \quad x_2 > 0, \\ 0, & otherwise, \end{cases} \qquad T_2 = \begin{cases} e^{-x_2}, & x_1 > -x_2, \quad x_2 < 0, \\ 0, & otherwise. \end{cases}$$

Then we have $T_1 \in H'_{\overline{\Gamma}}(\mathbb{R}^2, \{(0,1)\})$, $T_2 \in H'_{\overline{\Gamma}}(\mathbb{R}^2, \{(0,-1)\})$ and

$$\langle T_{1_x}, e^{\imath\zeta x}\rangle = f_1(\zeta), \quad \langle T_{2_x}, e^{\imath\zeta x}\rangle = f_2(\zeta), \quad T = T_1 + T_2.$$



6 Edge-of-the-Wedge theorem

In this section we shall see the Edge-of-the-Wedge theorem for the space of the image by the Fourier-Laplace transform of $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. First we introduce some spaces of holomorphic functions. For details we refer the reader to [10], [11].

Definition 6.1. For a subset A of $\mathbb{R}^n$, we define a set $\mathcal{T}(A)$ by $\mathcal{T}(A) = \mathbb{R}^n \times \imath A$.

Definition 6.2. For a convex compact set K of $\mathbb{R}^n$ and $\varepsilon > 0$,

$$\begin{aligned} &\mathcal{Q}_b(\mathcal{T}(K_\varepsilon)) \\ &\quad := \{\varphi(\zeta) \in \mathcal{H}(\mathcal{T}(K_\varepsilon^\circ)) \cap \mathcal{C}(\mathcal{T}(K_\varepsilon));\ \sup_{\zeta \in \mathcal{T}(K_\varepsilon)} |\zeta^\alpha \varphi(\zeta)| < \infty \text{ for } \forall \alpha \in \mathbb{N}^n\}, \end{aligned}$$

$$\mathcal{Q}(\mathcal{T}(K)) := \varinjlim_{\varepsilon > 0} \mathcal{Q}_b(\mathcal{T}(K_\varepsilon)).$$

Definition 6.3. The dual space $\mathcal{Q}'(\mathcal{T}(K))$ of $\mathcal{Q}(\mathcal{T}(K))$ is called tempered ultrahyperfunctions [10], [11].

We have the following theorem for the spaces $H(\mathbb{R}^n, K)$ and $\mathcal{Q}(\mathcal{T}(K))$:

Theorem 6.4 ([10]). *Let $\varphi(x) \in H(\mathbb{R}^n, K)$. The Fourier inverse transform*

$$\mathcal{F}^{-1}(\varphi)(\zeta) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \varphi(x) e^{-\imath\zeta x} dx$$

establishes a topological isomorphism of $H(\mathbb{R}^n, K)$ onto $\mathcal{Q}(\mathcal{T}(K))$. The inverse mapping $\mathcal{F}$ is given by

$$\mathcal{F}(\psi)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \psi(\xi + \imath\eta) e^{\imath(\xi+\imath\eta)x} d\xi, \quad \eta \in K_\varepsilon^\circ, \quad \psi \in \mathcal{Q}_b(\mathcal{T}(K_\varepsilon)). \tag{14}$$

Remark 6.5. In (14), we notice that $\mathcal{F}(\psi)(x)$ is independent of $\eta \in K_\varepsilon^\circ$ by Cauchy's integral theorem.

Definition 6.6 ([10]). For $T \in H'(\mathbb{R}^n, K)$, we define the dual Fourier transform $\mathcal{F}(T)$ as a continuous linear functional on $\mathcal{Q}(\mathcal{T}(K))$ by the formula

$$\langle \mathcal{F}(T), \psi \rangle = \langle T, \mathcal{F}(\psi) \rangle, \quad \text{for } \psi \in \mathcal{Q}(\mathcal{T}(K)). \tag{15}$$

As a consequence of Theorem 6.4, we have the following theorem:

Theorem 6.7 ([10]). *The dual Fourier transform* (15) *gives topological isomorphisms*

$$\mathcal{F} : H'(\mathbb{R}^n, K) \to \mathcal{Q}'(\mathcal{T}(K)).$$

Definition 6.8. Let $K = \{u\}$, $\psi \in \mathcal{Q}_b(\mathcal{T}(K_{\varepsilon_1}))$ and assume that $f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L)$ satisfies

$$\forall \varepsilon > 0\ \exists l_\varepsilon \geq 0\ \forall \overline{\Gamma}_C \Subset (\overline{\Gamma}')^\circ\ \exists M_{\varepsilon,\overline{\Gamma}_C} \geq 0\ s.t.$$

$$|f(\zeta)| \leq M(1 + |\zeta|)^l, \quad \zeta \in \mathbb{R}^n + \imath[\overline{\Gamma}_C]_{-\varepsilon}.$$

Then we define $\langle f(\zeta), \psi(\zeta) \rangle$ by

$$\begin{aligned} \langle f(\zeta), \psi(\zeta) \rangle &:= \langle f(\xi + \imath\eta_0), \psi(\xi + \imath\eta_0) \rangle \\ &= \int_{\mathbb{R}^n} f(\xi + \imath\eta_0)\psi(\xi + \imath\eta_0) d\xi, \end{aligned}$$

where $\eta_0 \in (\{u\} + (\overline{\Gamma}')^\circ) \cap (K_{\varepsilon_1}{}^\circ)$.

Definition 6.9. Let $K = \{u\}$, $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ and $\psi \in \mathcal{Q}(\mathcal{T}(K))$, $\psi \in \mathcal{Q}_b(\mathcal{T}(K_{\varepsilon_1}))$. By Theorem 5.5 and Definition 6.8, we define $\langle \mathcal{LF}(T)(\zeta), \psi(\zeta)\rangle$ by

$$\langle \mathcal{LF}(T)(\zeta), \psi(\zeta)\rangle := \langle \mathcal{LF}(T)(\xi + \imath\eta_0), \psi(\xi + \imath\eta_0)\rangle, \tag{16}$$

where $\eta_0 \in (\{u\} + (\overline{\Gamma}')^\circ) \cap (K_{\varepsilon_1}{}^\circ)$.

Now we can show Edge-of-the-Wedge theorem. For the direct product case, it is given in [11].

Theorem 6.10 (Edge-of-the-Wedge Theorem [14]). *Let Γ_1, Γ_2 be proper open convex cones in $\mathbb{R}^n$,*

$$L_m = \{u_m\} + (\overline{\Gamma}'_m)^\circ, \quad m = 1, 2.$$

Assume that $F_1(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L_1)$ and $F_2(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L_2)$ satisfy

$$\forall\varepsilon > 0\ \exists l_{m_\varepsilon} \geq 0\ \forall\overline{\Gamma}_{C_m} \Subset (\overline{\Gamma}'_m)^\circ\ \exists M_{\varepsilon,\overline{\Gamma}_{C_m}} \geq 0 \ s.t.$$

$$|F_m(\zeta)| \leq M_{\varepsilon,\overline{\Gamma}_{C_m}}(1 + |\zeta|)^{l_{m_\varepsilon}}, \quad \zeta \in \mathbb{R}^n + \imath[\overline{\Gamma}_{C_m}]_{-2\varepsilon}, \quad m = 1, 2, \tag{17}$$

where $[\overline{\Gamma}_{C_m}]_{-\varepsilon} = \mathbb{R}^n \backslash (\mathbb{R}^n \backslash (\{u_m\} + \overline{\Gamma}_{C_m})^\circ)_\varepsilon$.

Let K be a convex compact set which contains the segment with $\{u_1\}$ and $\{u_2\}$ as extremal point. Assume that

$$\langle F_1(\zeta), \psi(\zeta)\rangle = \langle F_2(\zeta), \psi(\zeta)\rangle \quad \forall\psi(\zeta) \in \mathcal{Q}(\mathcal{T}(K)). \tag{18}$$

Then there exists $F(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath(L'_1 \cup L'_2))$ such that

$$F(\zeta)|_{(\mathbb{R}^n + \imath L_1)} = F_1(\zeta), \quad F(\zeta)|_{(\mathbb{R}^n + \imath L_2)} = F_2(\zeta),$$

where $L'_1 = \{u_1\} + (\overline{\Gamma}'_1 \cup \overline{\Gamma}'_2)^\circ$ and $L'_2 = \{u_2\} + (\overline{\Gamma}'_1 \cup \overline{\Gamma}'_2)^\circ$. Furthermore

(i) *if $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \{0\}$, then $F(\zeta)$ is polynomial,*

(ii) *if $\{u_1\} = \{u_2\} (=: \{u\})$, then we have.*

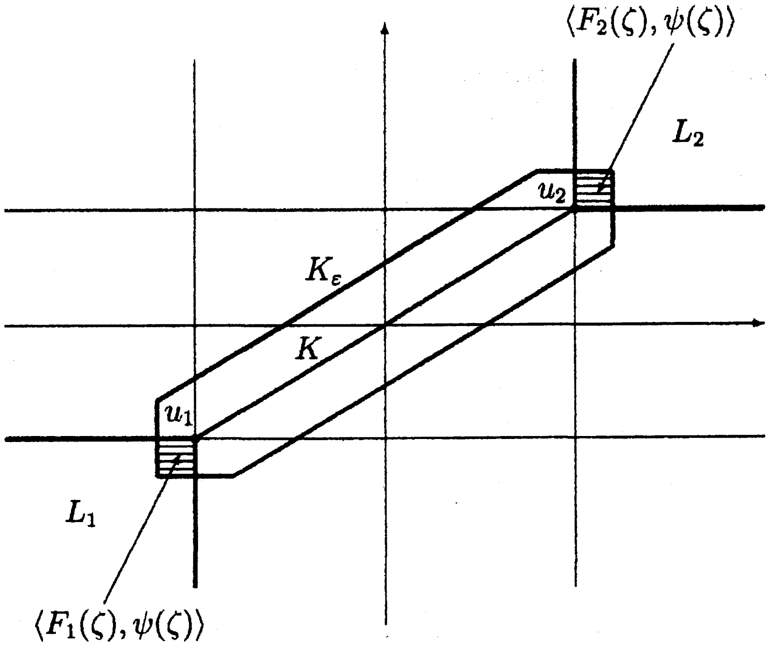
$$F(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath(\{u\} + (\overline{\Gamma}'_1 \cup \overline{\Gamma}'_2)^\circ)) \tag{19}$$

and

$$\forall\varepsilon > 0\ \exists l_\varepsilon \geq 0\ \forall\overline{\Gamma}_C \Subset (\overline{\Gamma}'_1 \cup \overline{\Gamma}'_2)^\circ\ \exists M_{\varepsilon,\overline{\Gamma}_C} \geq 0$$

$$|F(\zeta)| \leq M(1 + |\zeta|)^l, \quad \zeta \in \mathbb{R}^n + \imath[\overline{\Gamma}_C]_{-\varepsilon}, \tag{20}$$

where $[\overline{\Gamma}_C]_{-\varepsilon} = \mathbb{R}^n \backslash (\mathbb{R}^n \backslash (\{u\} + \overline{\Gamma}_C)^\circ)_\varepsilon$.



Acknowledgment: The author expresses his thanks to Professor K.Yoshino for many valuable suggestions.

References

[1] S.-Y.Chung and D.Kim : *Distributions with exponential growth and Bochner-Schwartz theorem for Fourier hyperfunctions*, Publ. RIMS. Kyoto Univ. 31, (1995), 829-845.

[2] C.Dong and T.Matsuzawa : *S-spaces of Gel'fand-Shilov and differential equations*, Japan. J. Math. Vol.19, No.2, 1994.

[3] M.Hasumi : *Note on the n-dimensional ultradistributions*, Tôhoku Math. J. 13, (1961), 94-104.

[4] L.Hörmander : *The Analysis of Linear Partial Differntial Operators I*, Springer-Verlag, (1983).

[5] L.Hörmander : *Notions of Convexity*, Birkhäuser, (1994).

[6] G.Lysik : *On the structure of Mellin distributions*, Ann. Polon. Math. 51, (1990), 219-228.

[7] T.Matsuzawa : *A calculus approach to the hyperfunctions I*, Nagoya Math. J. Vol.108, (1987), 53-66.

[8] T.Matsuzawa : *A calculus approach to the hyperfunctions II*, Trans. Amer. Math. Soc. 313 (1989), 619-654.

[9] T.Matsuzawa : *An introduction to the theory of partial differential equations*, JSPS-DOST Lecture Notes in Mathematics, Vol.4, (1997).

[10] M.Morimoto : *Theory of tempered ultrahyperfunctions I,II*, Proc. Japan Acad. Vol.51, (1975), 87-91, 213-218.

[11] M.Morimoto : *Convolutors for ultrahyperfunctions*, Lecture Notes in Physics, 39, Springer-Verlag, (1975).

[12] A.U.Schmidt *Asymptotische Hyperfunktionen, temperierte Hyperfunktionen und asymptotische Entwicklungen*, Logos Verlag Berlin, 1999.

[13] S.e.Silva : *Les fonctions analytiques commes ultra-distributions dans le calcul opérationnel*, Math.Ann. 136, (1958), 58-96.

[14] M.Suwa : *Distributions of exponential growth with support in a proper convex cone*, accepted for publication in Publ. RIMS, Kyoto Univ..

[15] S.Szmydt and B.Ziemian : *Laplace distributions and hyperfunctions on* $\overline{\mathbb{R}}^n_+$, J. Math. Sci. Univ. Tokyo 5, (1998), 41-74.

[16] V.S.Vladimirov : *Generalized Functions in Mathematical Physics*, Nauka Moscow, (1976).

[17] Z.Zieleźny : *On the space of convolution operators in* $\mathcal{K}_1'$, Studia Math., 31, 111-124, (1968).